IV. Conclusions

In this Note steady-state covariance matrices have been derived for the discrete ECA track filter when the measurement matrix is composed of position and velocity measurements. The MacFarlane–Potter–Fath eigenstructure method gives an answer to the discrete ECA filter in an elegant analytic fashion.

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Robust Controller Design with Damping and Stability Specifications

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Introduction

IN many physical systems, controllers must be designed to operate within a nominal domain that covers different stages of operation. Multiple models or a model with parametric uncertainties must be established to represent the dynamics. In dealing with systems characterized by multiple models, there are two methods for designing controllers by state feedback: 1) controllers based on pole assignment[1,2] and 2) controllers based on linear quadratic design.[2,3] Both techniques are well suited for tradeoffs between eigenvalue locations and requirements of robustness against model changes. In dealing with systems characterized by models with parametric uncertainties, robust controller design has gained new interest. By use of the Riccati-equation approach,[4,5] robust controllers have been proposed to ensure the stability of the overall system for all admissible parametric uncertainties. In this Note, the Riccati-equation approach is extended to design robust controllers with damping and stability characteristics.

Statement of the Problem

The linear systems described by the following dynamic equation are considered:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \left[A_0 + \sum_{i=1}^{p} r_i(t)\Delta A_i\right]x(t) + \left[B_0 + \sum_{i=1}^{p} r_i(t)\Delta B_i\right]u(t) \tag{1}$$

where $x \in \Re^n$ and $u \in \Re^m$, the nominal system matrix A_0 and the nominal input connection matrix B_0 are controllable, and each of the parametric uncertainties is modeled by an uncertain variable $r_i(t) \in \Re$ along with given constant matrices ΔA_i and ΔB_i. We assume that all uncertain variables are bounded for all time t such that

$$|r_i(t)| \leq 1 \qquad i = 1, 2, \ldots, p \tag{2}$$

Then, matrices ΔA_i and ΔB_i designate the ranges of deviation of parametric uncertainties.

For systems with state-space models, linear feedback control laws can be written as

$$u(t) = -Kx(t) \tag{3}$$

Thus, closed-loop dynamics are

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = A_c x(t)$$

$$A_c = A_0 - B_0 K + \sum_{i=1}^{p} r_i(t)\{\Delta A_i - \Delta B_i K\} \tag{4}$$

In the following, we denote the closed-loop eigenvalues by

$$\lambda_k(A_c) \qquad k = 1, 2, \ldots, n \tag{5}$$

The eigenvalues are located within the design sector $D(\alpha, \theta)$ in the complex plane if, for a particular choice of real positive numbers α and θ, we have

$$\mathrm{Re}(\lambda_k) < -\alpha \qquad \alpha \geq 0 \tag{6a}$$

$$|\mathrm{Im}(\lambda_k)|\tan(\theta) < -\mathrm{Re}(\lambda_k) - \alpha \qquad \theta \in [0, \pi/2) \tag{6b}$$

Our problem is to determine linear feedback control laws (3) for uncertain linear systems (1) such that, for all admissible uncertainties, the closed-loop poles (5) are located within the design sector $D(\alpha, \theta)$.

The design sector $D(\alpha, \theta)$ has been presented in the following well-known theorem.[6]

Relative Stability Theorem: Given matrix $A_c \in \Re^{n\times n}$, $\lambda_k(A_c)$ lie within $D(\alpha, \theta)$ if and only if the eigenvalues of matrix $H(\theta) \otimes (A_c + \alpha I) \in \Re^{2n\times 2n}$ lie in the left-half complex plane (see Ref. 6 for proof). Here, the matrix $H(\theta) \in \Re^{2\times 2}$ is given by

$$H(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \tag{7}$$

and the $\otimes$ denotes the Kronecker product such that

$$H(\theta) \otimes (A_c + \alpha I) = \begin{bmatrix} \cos(\theta)(A_c + \alpha I) & -\sin(\theta)(A_c + \alpha I) \\ \sin(\theta)(A_c + \alpha I) & \cos(\theta)(A_c + \alpha I) \end{bmatrix} \tag{8}$$

In this Note, a new formulation of the Riccati equation is proposed. The robust controller design is formulated as a linear quadratic state feedback problem with prescribed damping and stability characteristics. Thus, given the bounds of the system parameters, the proposed controller can ensure that the overall system is asymptotically stable with a prescribed degree of damping and stability for all admissible parametric uncertainties.

Method of Controller Design

In this section, the design of robust controllers involves the determination of matrices P, $Q > 0$ (i.e., matrices P and Q are positive definite) such that the following Riccati equation is fulfilled:

$$(A_0 + \alpha I)^T P + P(A_0 + \alpha I) - PB_0(R/2)^{-1}B_0^T P + 2Q = 0 \tag{9}$$

where matrix $R > 0$ is chosen by the designer. The following result can be obtained.

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Theorem 1: All closed-loop eigenvalues (5) of the linear system (1) lie within the design sector $D(\alpha, \theta)$ via the linear feedback control laws (3) with

$$K = R^{-1} B_0^T P \tag{10}$$

if there exist matrices P, $Q > 0$ for Eq. (9) such that

$$\begin{bmatrix} \cos(\theta) Q & \sin(\theta)\Pi_1^T \\ \sin(\theta)\Pi_1 & \cos(\theta) Q \end{bmatrix} > \sum_{i=1}^{p} \begin{bmatrix} \cos(\theta)\Pi_{2i} & \sin(\theta)\Pi_{3i}^T \\ \sin(\theta)\Pi_{3i} & \cos(\theta)\Pi_{2i} \end{bmatrix}_{ps} \tag{11}$$

where

$$\Pi_1 = -[P A_0]_{sk} \tag{12a}$$

$$\Pi_{2i} = \left[P \Delta A_i - P \Delta B_i R^{-1} B_0^T P\right]_s \tag{12b}$$

$$\Pi_{3i} = \left[P \Delta A_i - P \Delta B_i R^{-1} B_0^T P\right]_{sk} \tag{12c}$$

Here, $[\,]_s$ ($[\,]_{sk}$) denotes the symmetric (skew-symmetric) portion of a square matrix $[\,]$ and $[\,]_{ps}$ denotes a positive semidefinite matrix formed by use of the following operations: 1) take the symmetric portion of a square matrix $[\,]$ and 2) replace each of the eigenvalues of the symmetric matrix by its modulus value. Furthermore, if it is required that the controller is designed to guarantee only a prescribed degree of stability (i.e., $\theta = 0$), condition (11) is reduced to the following simpler relation:

$$Q > \sum_{i=1}^{p} [\Pi_{2i}]_{ps} \tag{13}$$

Proof: A simple computation shows that

$$-2\begin{bmatrix} \cos(\theta) Q & \sin(\theta)\Pi_1^T \\ \sin(\theta)\Pi_1 & \cos(\theta) Q \end{bmatrix} + 2\sum_{i=1}^{p} r_i(t) \begin{bmatrix} \cos(\theta)\Pi_{2i} & \sin(\theta)\Pi_{3i}^T \\ \sin(\theta)\Pi_{3i} & \cos(\theta)\Pi_{2i} \end{bmatrix}$$
$$= [H(\theta) \otimes (A_c + \alpha I)]^T [I_2 \otimes P] + [I_2 \otimes P][H(\theta) \otimes (A_c + \alpha I)] \tag{14}$$

Then, using the Lyapunov stability theory and the relative stability theorem, $\lambda_k(A_c)$ lie within $D(\alpha, \theta)$ if the following condition is fulfilled:

$$\begin{bmatrix} \cos(\theta) Q & \sin(\theta)\Pi_1^T \\ \sin(\theta)\Pi_1 & \cos(\theta) Q \end{bmatrix} > \sum_{i=i}^{p} r_i(t) \begin{bmatrix} \cos(\theta)\Pi_{2i} & \sin(\theta)\Pi_{3i}^T \\ \sin(\theta)\Pi_{3i} & \cos(\theta)\Pi_{2i} \end{bmatrix} \tag{15}$$

It is known that, for all symmetric matrices $[\,]$, we have $[\,]_{ps} \geq r_i(t)[\,]$. Therefore, condition (11) is sufficient for justifying that $\lambda_k(A_c)$ lie within $D(\alpha, \theta)$. □

Define matrices Π_4, Π_5, and $\Pi_6 \in \Re^{n \times n}$ by

$$\begin{bmatrix} \Pi_4 & \Pi_5^T \\ \Pi_5 & \Pi_4 \end{bmatrix} \equiv \sum_{i=1}^{p} \begin{bmatrix} \cos(\theta)\Pi_{2i} & \sin(\theta)\Pi_{3i}^T \\ \sin(\theta)\Pi_{3i} & \cos(\theta)\Pi_{2i} \end{bmatrix}_{ps}$$
$$\Pi_6 \equiv \sin(\theta)\Pi_1 - \Pi_5 \tag{16}$$

The following corollary is used to propose a design procedure for obtaining the robust controllers.

Corollary 1: All closed-loop eigenvalues (5) of the linear system (1) lie within $D(\alpha, \theta)$ via the controller (10), if there exists solution $P > 0$ to the Riccati equation (9) with Q given by

$$Q = \varepsilon I + \frac{\bar{Q}}{\cos(\theta)}$$
$$\bar{Q} = \Pi_4 + \left(\Pi_6^T \Pi_6\right)^{\frac{1}{2}} + \left[\left(\Pi_6^T \Pi_6\right)^{\frac{1}{2}} - \left(\Pi_6 \Pi_6^T\right)^{\frac{1}{2}}\right]_{p0} \tag{17}$$

where $\varepsilon > 0$ is a small constant. Here, we use the notation $[\,]_{p0}$ to denote a positive semidefinite matrix formed by use of the following operations: 1) take the symmetric portion of a square matrix $[\,]$, 2) replace each of the positive eigenvalues of the symmetric matrix by zero, and 3) replace each of the eigenvalues of the symmetric matrix by its modulus value. Furthermore, if it is required that the controller is designed to guarantee only a prescribed degree of stability (i.e., $\theta = 0$), condition (17) is reduced to the following simpler relation:

$$Q = \varepsilon I + \sum_{i=1}^{p} [\Pi_{2i}]_{ps} \tag{18}$$

Proof: Following from Eq. (17), we have

$$\cos(\theta) Q - \Pi_4 > \left(\Pi_6^T \Pi_6\right)^{\frac{1}{2}} + \left[\left(\Pi_6^T \Pi_6\right)^{\frac{1}{2}} - \left(\Pi_6 \Pi_6^T\right)^{\frac{1}{2}}\right]_{p0} \tag{19}$$

The definition of the matrix $[\,]_{p0}$ results in the following relations:

$$\cos(\theta) Q - \Pi_4 > \left(\Pi_6^T \Pi_6\right)^{\frac{1}{2}}$$
$$\cos(\theta) Q - \Pi_4 > \left(\Pi_6 \Pi_6^T\right)^{\frac{1}{2}} \tag{20a}$$

$$\begin{bmatrix} \cos(\theta) Q - \Pi_4 & \sin(\theta)\Pi_1^T - \Pi_5^T \\ \sin(\theta)\Pi_1 - \Pi_5 & \cos(\theta) Q - \Pi_4 \end{bmatrix} > \begin{bmatrix} \left(\Pi_6^T \Pi_6\right)^{\frac{1}{2}} & \Pi_6^T \\ \Pi_6 & \left(\Pi_6 \Pi_6^T\right)^{\frac{1}{2}} \end{bmatrix} \tag{20b}$$

Thus, the condition (11) of Theorem 1 will be fulfilled if

$$\begin{bmatrix} \left(\Pi_6^T \Pi_6\right)^{\frac{1}{2}} & \Pi_6^T \\ \Pi_6 & \left(\Pi_6 \Pi_6^T\right)^{\frac{1}{2}} \end{bmatrix} \geq 0 \tag{21}$$

Denote the singular value decomposition of matrix Π_6 by

$$\Pi_6 = U \Sigma V^T \tag{22}$$

Then,

$$\begin{bmatrix} \left(\Pi_6^T \Pi_6\right)^{\frac{1}{2}} & \Pi_6^T \\ \Pi_6 & \left(\Pi_6 \Pi_6^T\right)^{\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} V \Sigma^{\frac{1}{2}} \\ U \Sigma^{\frac{1}{2}} \end{bmatrix} \left[\Sigma^{\frac{1}{2}} V^T \quad \Sigma^{\frac{1}{2}} U^T\right] \geq 0 \tag{23}$$

Using Theorem 1 and relation (20), $\lambda_k(A_c)$ lie within the design sector $D(\alpha, \theta)$. □

Now, our method of robust controller design is as follows:

1) Find the range of the system parameters.

2) Choose the nominal parameters for the system, and decide the maximum ranges of deviation of the uncertainties.

3) Make ε small [e.g., $\varepsilon = 0.01$ for Eq. (18)], and choose a design sector $D(\alpha, \theta)$ and a constant matrix R.

4) Use the algorithm given after step 5 to obtain the solution P for Eq. (9) such that condition (11) is fulfilled.

5) Construct the linear state feedback controller using Eq. (10).

The algorithm for solving Eq. (9) with condition (11) is as follows.

1) Solve the Riccati equation (9) for P_0 where $Q = \varepsilon I$ is assigned. The matrix P_0 is used to initialize an iterative procedure (i.e., steps 2–4).

2) Given matrix P_j, the following matrices are determined:

$$\Pi_1 = -[P_j A_0]_{sk} \tag{24a}$$

$$\Pi_{2i} = \left[P_j \Delta A_i - P_j \Delta B_i R^{-1} B_0^T P_j\right]_s$$
$$\Pi_{3i} = \left[P_j \Delta A_i - P_j \Delta B_i R^{-1} B_0^T P_j\right]_{sk} \tag{24b}$$

$$\begin{bmatrix} \Pi_4 & \Pi_5^T \\ \Pi_5 & \Pi_4 \end{bmatrix} \equiv \sum_{i=1}^{p} \begin{bmatrix} \cos(\theta)\Pi_{2i} & \sin(\theta)\Pi_{3i}^T \\ \sin(\theta)\Pi_{3i} & \cos(\theta)\Pi_{2i} \end{bmatrix}_{ps} \tag{24c}$$

$$Q_j = \left[P_j B_0 R^{-1} B_0^T P_j - P_j (A_0 + \alpha I)\right]_s \tag{24d}$$

$$\Pi = \begin{bmatrix} \cos(\theta) Q_j - \Pi_4 & \sin(\theta)\Pi_1^T - \Pi_5^T \\ \sin(\theta)\Pi_1 - \Pi_5 & \cos(\theta) Q_j - \Pi_4 \end{bmatrix}$$
$$\Pi_6 \equiv \sin(\theta)\Pi_1 - \Pi_5 \tag{24e}$$

3) If $\Pi \geq 0$, stop and declare that the solution $P = P_j$ and $Q = Q_j$ fulfills Eq. (9) with condition (11). Otherwise, matrix Q is determined by

$$Q = \varepsilon I + \frac{\bar{Q}}{\cos(\theta)}$$
$$\bar{Q} = \Pi_4 + \left(\Pi_6^T \Pi_6\right)^{\frac{1}{2}} + \left[\left(\Pi_6^T \Pi_6\right)^{\frac{1}{2}} - \left(\Pi_6 \Pi_6^T\right)^{\frac{1}{2}}\right]_{p0} \tag{25}$$

Using Q, a standard algorithm is applied to solve the following equation for P_{j+1}:

$$\left[A_0 + \alpha I - B_0 (2R)^{-1} B_0^T P_j\right]^T P_{j+1} + P_{j+1}\left[A_0 + \alpha I - B_0(2R)^{-1} B_0^T P_j\right] - P_{j+1} B_0 R^{-1} B_0^T P_{j+1} + 2Q = 0 \tag{26}$$

4) If the norm of matrix $P_{j+1} - P_j$ is less than some computational accuracy, stop and declare that the algorithm converges to the solution $P = P_j$ and $Q = Q_j$ that fulfills Eq. (9) with condition (11). Otherwise, repeat from step 2 using matrix P_{j+1}.

Remark 1: The inclusion of the small parameter ε guarantees that a unique positive definite solution exists for Eq. (26). However, the algorithm may not converge if the parametric uncertainties of the system are excessively large or if too large a prescribed degree of damping ($\sin\theta$) and stability (α) is selected.

Examples

In this section, the proposed design method is applied to the control of uncertain mass/spring systems. Uncertainties in the open-loop system matrix will be assumed. This is representative of a structural system where mode frequencies and damping values are unknown. Our approach assumes full state feedback. Though this may not be realistic in real systems, understanding the design results will be helpful when we assume knowledge of only the output variables.

Example 1: Consider a two-mass/spring system characterized by

$$A_0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$
$$r_1(t)\Delta A_1 = 0.4 \times \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -r_1 & r_1 & 0 & 0 \\ r_1 & -r_1 & 0 & 0 \end{bmatrix} \tag{27a}$$

$$B_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \qquad r_1(t)\Delta B_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{27b}$$

Apply the proposed design procedure with $\varepsilon = 0.01$, $R = 1$, and $D(\alpha = 0.2, \sin\theta = 0.2)$; the robust controller is obtained as

$$K = [26.44 \quad -19.01 \quad 6.290 \quad 25.89] \tag{28}$$

It is noted that the uncertainty in this system is contained solely in the potential energy of the spring. Although the system is initially displaced from its equilibrium position, the control brought the spring swiftly to its equilibrium length so as to keep the uncertainty potential energy from adversely affecting the dynamics of the motion. Furthermore, the overall system exhibits nearly the same transient responses for all admissible uncertainties (r_1).

Example 2: Consider a four-mass/spring system characterized by

$$A_0 = \begin{bmatrix} 0_4 & I_4 \\ \hat{A} & 0_4 \end{bmatrix}, \qquad B_0 = \begin{bmatrix} 0_4 \\ \hat{B} \end{bmatrix}$$
$$\hat{A} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \qquad \hat{B} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{29a}$$

$$\sum_{i=1}^{3} r_i(t)\Delta A_i = 0.4 \times \begin{bmatrix} 0_4 & 0_4 \\ \sum_{i=1}^{3} r_i(t)\Delta \hat{A}_i & 0_4 \end{bmatrix}$$
$$\sum_{i=1}^{3} r_i(t)\Delta B_i = \begin{bmatrix} 0_2 \\ 0_2 \\ 0_2 \\ 0_2 \end{bmatrix} \tag{29b}$$

$$\sum_{i=1}^{3} r_i(t)\Delta \hat{A}_i = \begin{bmatrix} -r_1 & r_1 & 0 & 0 \\ r_1 & -r_1 - r_2 & r_2 & 0 \\ 0 & r_2 & -r_2 - r_3 & r_3 \\ 0 & 0 & r_3 & -r_3 \end{bmatrix} \tag{29c}$$

Apply the proposed design procedure with $\varepsilon = 0.01$, $R = 1$, and $D(\alpha = 0.2, \sin\theta = 0.2)$; the robust controller is obtained as

$$K = \begin{bmatrix} -27.34 & 44.59 & -2.759 & -2.853 & 52.67 & 8.456 & -0.437 & -9.536 \\ -2.853 & -2.729 & 44.59 & -27.34 & -9.536 & -0.437 & 8.456 & 52.67 \end{bmatrix} \tag{30}$$

Although the system is initially displaced from its equilibrium position, it is noted that the control worked to transfer almost all spring potential energy to the second spring. Because the length of the second spring is firmly governed via control inputs, the uncertain potential energy will not adversely affect the dynamics of the motion.

Conclusions

A method of designing robust controllers for linear systems with parametric uncertainties has been presented. The obtained control laws can guarantee that the control systems are asymptotically stable with prescribed degree of damping and stability for all admissible uncertainties. Two examples of mass/spring systems with uncertain spring constants have been considered. It has been shown that the key to the robust performance of the proposed design is a controller that can keep the uncertainty energy of the spring from adversely affecting the dynamics of the motion.

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Nonlinear Dynamics of the Tethered Subsatellite System in the Station Keeping Phase

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I. Introduction

THE tethered subsatellite system[1–5] (TSS) mission requires three phases: deployment, station keeping, and retrieval. Let us focus on the analysis of the nonlinear dynamics for the TSS in the station keeping phase, where tether is assumed to have constant natural length. Even in the station keeping phase, TSS motion shows highly nonlinear behavior taking into account complex effects caused by such perturbation parameters as the orbit eccentricity and the tether elasticity. Nonlinear dynamics is investigated in the literature[6–11] because the dynamics is not easily analyzed and exhibits interesting behavior such as chaos. Nonlinear analysis tools are employed in this study, such as Poincaré maps, bifurcation diagrams, and Lyapunov exponents, and show periodic, quasiperiodic, and chaotic motion, depending on each system parameter. Kanasopoulos and Richardson[6,7] investigated the nonlinear dynamics of a gravity-gradient satellite, and Nixon and Misra[8] did so for the TSS with a rigid tether. Our objective is not just an application of the nonlinear analysis but to point out the importance of the nonlinear analysis of TSS by taking into account the complex effects of its system parameters.

II. Equations of Motion

The TSS model treated in this paper is a gravity-gradient tethered subsatellite in an elliptical orbit. Eccentricity and the longitudinal rigidity of tether are chosen for the system parameters. Energy dissipation such as the aerodynamic effect will be ignored in this system. The dynamical model is simplified by employing the following assumptions.

1) The Shuttle and subsatellite are point masses.
2) The center of mass of the system coincides with that of the Shuttle.
3) Tether length is sufficiently smaller than the distance between the Shuttle and the center of the Earth.
4) The tether is approximately regarded as a linear spring whose natural length is 100 km, and its mass is negligible.

The equations of motion for TSS for two cases are described as follows.

1) The first case is that of a circular orbit, and the tether has no elasticity:

$$\ddot{\theta} = -\tfrac{3}{2}\sin(2\theta) \tag{1}$$

2) The second case is that of an elliptical orbit,[12] and the tether has elasticity[4] in the pitch rotation direction,

$$l(1 + e\cos\eta)\ddot{\theta} - 2el(\dot{\theta} + 1)\sin\eta + \tfrac{3}{2}l\sin 2\theta + 2(1 + e\cos\eta)(\dot{\theta} + 1)\dot{l} = 0 \tag{2}$$

and in the tether length direction,

$$(1 + e\cos\eta)\ddot{l} - 2e\sin\eta\dot{l} - l(1 + e\cos\eta)(2 + \dot{\theta})\dot{\theta} - 3l(1 + e\cos\eta)\cos^2\theta + (T/m)(R^3/\mu) = 0 \tag{3}$$

l, θ, l_0, m, a, e, μ, and EA denote tether length, pitch rotation, natural length of tether, mass of subsatellite, semimajor radius (which is constant at 6600 km in this case), eccentricity of orbit, gravitational constant, and the longitudinal rigidity of tether, respectively. The tension of the tether T and the orbit radius R are defined as $T = EA(l - l_0)/l_0$ and $R = a(1 - e^2)/(1 + e\cos\eta)$, respectively, where η denotes true anomaly and indicates the independent variable.

III. Chaos

Chaos is introduced here to recognize the employed nonlinear analysis. Chaotic motion means that the behavior of the system can be predicted for the short term but cannot for all of the time, and the motion is regarded as chaotic if it simultaneously satisfies the following two properties[11]: sensitive dependence on the initial conditions and topological transition. When the equations of motion are nonlinear and do not have analytical solutions, the motion becomes chaotic over some range of the initial conditions or system parameters. Concerning TSS, two system parameters, eccentricity and the longitudinal rigidity of tether, can affect the nonlinear dynamics through the coefficients of nonlinear terms. Therefore, chaotic motion may occur in the situation by taking either of them into account, as in Eqs. (2) and (3). On the other hand, chaotic motion never occurs in the situation without taking both of them into account, as in Eq. (1).

IV. Poincaré Maps

A Poincaré map is a collection of discrete plots created by integrating the equations of motion and periodically sampling the following particular points in the trajectory[6,7,10]:

$$\theta_n = \theta(\eta_n) \tag{4}$$

$$\dot{\theta}_n = \dot{\theta}(\eta_n) \tag{5}$$

where $\eta_n = \eta_0 + n\Delta\eta$ ($n = 0, 1, 2, \ldots, N$) and n is the number of orbits. Here, Poincaré maps are drawn by gathering these points until $N = 100$ once per orbit, where the subsatellite passes through perigee with many initial conditions. On the Poincaré map, periodic motion is shown as one or more fixed points, quasiperiodic motion as a closed curve if a sufficient number of points are plotted, and chaotic motion as a scattering of points.

A Poincaré map in the rigid-body tether system with constant eccentricity, $e = 0.0$, is shown in Fig. 1. The origin and $(\theta, \dot{\theta}) = (\pm\pi, 0)$ are stable equilibriums, and $(\theta, \dot{\theta}) = (\pm\pi/2, 0)$ are unstable equilibriums. In the stable equilibriums, the motion of the TSS has one period in a synchronous orbit and the tether directs toward the local vertical direction. In the unstable equilibriums, at a distance $\pi/2$ rad from stable equilibriums, the tether directs along the local horizontal. The TSS motion is separated into two parts by a separatrix passing through unstable equilibriums, that is, the motion is libration inside the separatrix and tumbling outside the separatrix. Perturbing at eccentricity, $e = 0.05$ as in Fig. 2, chaos occurs in the neighborhood of the unstable equilibriums.

V. Bifurcation Diagrams

A bifurcation diagram is constituted by sampling points of $\theta(\eta_n)$ in the trajectory, corresponding with Eq. (4) in the same way as for the Poincaré maps, and is plotted with respect to eccentricity for a stable equilibrium point.

Bifurcation diagrams are shown in Figs. 3 and 4 for the case of the rigid-body tether system and the elastic tether system for $EA = 10^4$ N, respectively. These diagrams clarify the process that a stable equilibrium, whose period is one per one orbit at the initial state, bifurcates to different equilibriums whose periods are not one per one orbit. The motion of period 3 appears at $e = 0.25$ and 0.20,

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